Four vector potentials : Transformation equations for the Electromagnetic potentials $\vec{A}$ and ϕ :

Electromagnetic potentials or Electrodynamic potentials

Magnetic vector potential $\vec{A}$ and scalar potential ϕ are combinedly known as electromagnetic potentials or Electrodynamic potentials.

* magnetic vector potential $\vec{A}$ is a generator potential of magnetic field $\vec{B}$.

As we know that $\vec{B} = \text{curl}\,\vec{A}$

Lorentz condition :

$$\text{div}\,\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \qquad (1)$$

Lorentz condition (1) is a relation between magnetic vector potential $\vec{A}$ and scalar potential ϕ.

Equation (1) also suggests that if we define a four vector potential A_μ then

$$A_\mu = (\vec{A}, \tfrac{i}{c}\phi) \qquad (2)$$

Here result (2) is directly written. Now we have to justify the result (2).

Justification:– Maxwell's field equation in terms of electro-magnetic potential $\vec{A}$ and ϕ can be written as

$$\Box^2 \vec{A} = -\mu_0 \vec{J}$$

or $$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}$$

or $$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \qquad (3)$$

and $$\Box^2 \phi = -\frac{\rho}{\varepsilon_0}$$

or $$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{\rho}{\varepsilon_0}$$

or $$\nabla^2 \phi - \frac{1}{c^2}\cdot\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \qquad (4)$$

Equations ③ and ④ can be written as

$$\nabla^2 A_1 - \frac{1}{c^2}\frac{\partial^2 A_1}{\partial t^2} = -\mu_0 J_1 \quad \text{——— 5(a)}$$

$$\nabla^2 A_2 - \frac{1}{c^2}\frac{\partial^2 A_2}{\partial t^2} = -\mu_0 J_2 \quad \text{——— 5(b)}$$

$$\nabla^2 A_3 - \frac{1}{c^2}\frac{\partial^2 A_3}{\partial t^2} = -\mu_0 J_3 \quad \text{——— 5(c)}$$

and $$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad \text{——— 5(d}_1\text{)}$$

Multiplying equ^n 5(d$_1$) bothsides by $\frac{i}{c}$, we get

$$\nabla^2\left(\frac{i\phi}{c}\right) - \frac{1}{c^2}\frac{\partial^2\left(\frac{i\phi}{c}\right)}{\partial t^2} = -\frac{i\rho}{c\epsilon_0}$$

$$\Rightarrow \nabla^2 A_4 - \frac{1}{c^2}\frac{\partial^2 A_4}{\partial t^2} = -i\rho\mu_0 c \qquad \because A_4 = \frac{i\phi}{c}$$

$$c^2 = \frac{1}{\mu_0\epsilon_0} \Rightarrow c\mu_0 = \frac{1}{c\epsilon_0}$$

$$\Rightarrow \nabla^2 A_4 - \frac{1}{c^2}\cdot\frac{\partial^2 A_4}{\partial t^2} = -\mu_0 J_4 \quad \text{——— 5(d}_2\text{)} \qquad \because J_4 = ic\rho.$$

Equ^{n}s 5(a), 5(b), 5(c) and 5(d$_2$) can combinedly be written as

$$\nabla^2 A_\mu - \frac{1}{c^2}\frac{\partial^2 A_\mu}{\partial t^2} = -\mu_0 J_\mu$$

If we put $\mu = 1, 2, 3$ and 4 then we get equ^{n}s 5(a), 5(b), 5(c) & 5(d).

$$\Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A_\mu = -\mu_0 J_\mu$$

$$\Rightarrow \boxed{\Box^2 A_\mu = -\mu_0 J_\mu} \quad \text{——— ⑥}$$

Here μ_0 = a scalar and J_μ = current four vector

Therefore the product $\mu_0 J_\mu$ will be four vector

ie, R.H.S. of equ^n ⑥ is a four vector

Since R.H.S of equ^n ⑥ is a four vector, so LHS should be four vector. Since D'Alembertian operator $\Box^2$ is Lorentz invariant ie, $\Box'^2 = \Box^2$.

For LHS to be a four vector, A_μ must be four vector.

This four vector A_μ is known as four vector potential. which is combined or compact form of $\vec{A}$ and ϕ.

Lorentz transformation of four vector potential A_μ

Since all four vectors follow the Lorentz transformation so four vector potential A_μ should follow the Lorentz transformation of four vector.

From Lorentz transformation of four vector,

$$A'_\mu = \alpha_{\mu\nu} \cdot A_\nu \quad \text{——— (7)}$$

where A'_μ = components of four vector potential in frame S'

A_ν = " " " " " S

and $\alpha_{\mu\nu}$ = Transformation matrix (4×4 square matrix)

$\mu = 1, 2, 3, 4$ and $\nu = 1, 2, 3, 4$.

$$\alpha_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

Equⁿ (7) can be written in matrix form as

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \times \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \quad \text{——— (8)}$$

On solving equⁿ (8), we get

$$A'_1 = \gamma \cdot A_1 + 0 \cdot A_2 + 0 \cdot A_3 + i\beta\gamma \cdot A_4 = \gamma(A_1 + i\beta A_4)$$

$$\Rightarrow A'_1 = \gamma\left(A_1 + i\frac{v}{c} \cdot \frac{i\phi}{c}\right) \qquad \because \beta = \frac{v}{c},\ A_4 = \frac{i\phi}{c}$$

$$\Rightarrow A'_1 = \gamma\left(A_1 - \frac{v}{c^2}\phi\right) \text{——— 9 (a)} \qquad \because i^2 = -1$$

Again $A'_2 = 0 \cdot A_1 + 1 \cdot A_2 + 0 \cdot A_3 + 0 \cdot A_4$

$$\Rightarrow A'_2 = A_2 \text{——— 9 (b)}$$

Again $A'_3 = 0 \cdot A_1 + 0 \cdot A_2 + 1 \cdot A_3 + 0 \cdot A_4$

$$\Rightarrow A'_3 = A_3 \text{——— 9 (c)}$$

Again $A_4' = -i\beta\gamma \cdot A_1 + 0 \cdot A_2 + 0 \cdot A_3 + \gamma \cdot A_4$

$$A_4' = \gamma(A_4 - i\beta A_1) \qquad \text{--- 9 (d}_1\text{)}$$

or $\dfrac{i\phi'}{c} = \gamma\left(\dfrac{i\phi}{c} - i\dfrac{v}{c}A_1\right)$ $\quad \because A_4' = \dfrac{i\phi'}{c},\ A_4 = \dfrac{i\phi}{c},\ \beta = \dfrac{v}{c}$

$$\Rightarrow \phi' = \gamma(\phi - vA_1) \qquad \text{--- 9 (d}_2\text{)}$$

Thus Lorentz transformation of four vector potential will be

$$\left.\begin{aligned} &A_1' = \gamma\left(A_1 - \frac{v}{c^2}\phi\right),\quad A_2' = A_2,\quad A_3' = A_3 \\ &\text{and } A_4' = \gamma(A_4 - i\beta A_1) \text{ or } \phi' = \gamma(\phi - vA_1) \end{aligned}\right\} \text{--- (9)}$$

* Inverse Lorentz transformation of 4-vector potential:

For obtaining inverse Lorentz transformation of 4-vector potential, we interchange primed and unprimed quantities and put $-v$ in place of v in eqn (9).

Inverse Lorentz transformation of 4-vector potential will be

$$\left.\begin{aligned} &A_1 = \gamma\left(A_1' + \frac{v}{c^2}\phi'\right),\quad A_2 = A_2',\quad A_3 = A_3' \\ &\text{and } A_4 = \gamma(A_4' + i\beta A_1') \text{ or } \phi = \gamma(\phi' + vA_1') \end{aligned}\right\} \text{--- (10)}$$

* Lorentz condition in terms of four vector potential:

Lorentz condition is $\operatorname{div}\vec{A} + \dfrac{1}{c^2}\dfrac{\partial\phi}{\partial t} = 0$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} + \frac{i/c}{ic}\cdot\frac{\partial\phi}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} + \frac{\partial\left(\frac{i\phi}{c}\right)}{\partial(ict)} = 0$$

$$\Rightarrow \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0 \qquad \because A_4 = \frac{i\phi}{c},\ x_4 = ict$$

$$\Rightarrow \boxed{\frac{\partial A_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \Box\cdot A_\mu = 0} \qquad \text{--- (11)}$$

eqn (11) represents Lorentz condition or Lorentz gauge condition in terms of four vector potential.

Box operator $\Box \equiv \dfrac{\partial}{\partial x_\mu}$ = 4 D divergence operator.